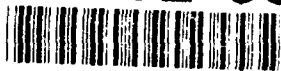


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A RAND NOTE

Monotone Polygon Intersection:
Geometry, Computer Applications,
and Computer Graphics

Darrell Harvey Shane

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A RAND NOTE

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Monotone Polygon Intersection: Geometry, Computer Applications, and Computer Graphics

Darrell Harvey Shane

Prepared for the
Defense Advanced Research Projects Agency

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PREFACE

This Note describes some results of ongoing research on techniques for managing geographic data by a database management system. The research is part of an effort to increase U.S. force effectiveness by improving the information that forces will use for planning, deploying, and executing orders through interactive electronic maps. Its objective is to support the consistent presentation of map and force information at different command levels, using maps of different resolutions.

This work is sponsored by the Defense Advanced Research Projects Agency. It is being carried out in the Information Processing Systems Program of the National Defense Research Institute, RAND's federally funded research and development center sponsored by the Office of the Secretary of Defense and the Joint Staff.

SUMMARY

Spatial queries are a fundamental class of queries on geographic data that consist primarily of points, lines, and polygons. Enhancing response time of spatial queries requires a spatial index.

The *chaintree* is a dynamic spatial index structure being developed at RAND, based on a regular planar straight line graph. The chaintree organizes polygons to efficiently answer spatial queries. An operation basic to most chaintree procedures is polygon intersection. This Note presents a new algorithm for finding the intersection of two uniformly monotone polygons in linear time and space without preprocessing.

ACKNOWLEDGMENT

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1. INTRODUCTION

The *chaintree*, a spatial index structure based on a decomposition of the plane into monotone subdivisions (Lee and Preparata, 1977; Preparata and Tamassia, 1989), was developed to address the problem of providing quick response to geographic vector queries requiring point location and polygon intersection operations.

This research has been conducted as part of an effort to develop an electronic environment for displaying, storing, accessing, and deriving different resolutions of geographic vector data. The various resolutions are derived electronically, using a knowledge base of descriptions, functions, and rules that express relationships describing how an object is to be represented and presented at varying resolution levels.

Interactive queries to an electronic map often involve geometric operations which can take an excessive amount of time if the geographic data are stored using more conventional indexing techniques (e.g., B-trees, R-trees). In particular, the two types of queries examined in this study are the *point location problem* and the *polygon intersection problem* (Cole and Yap, 1984). Given a query point p and a set of polygons \mathcal{P} , the answer to the point location problem is the set of polygons containing p . Given a query polygon P and a set of polygons \mathcal{P} , the set consisting of those polygons that intersect P form the answer to the polygon intersection problem. To enhance response time of these two types of spatial queries, we are developing the chaintree. An important function in this development process is the intersection of two monotone polygons.

Polygon intersection is a fundamental problem in computational geometry. It is well known that to compute the intersection of two simple (non-self-intersecting) polygons requires $\Theta(n^2)$ time, n being the total number of vertices. However, algorithms for finding the intersections of convex polygons, a special type of simple polygon, in linear time are well known (Shamos and Hoey, 1976; O'Rourke et al., 1982). The class of monotone polygons properly contains the class of convex polygons and naturally arises when taking the difference, union, or intersection of two convex polygons. Whereas the intersection of two convex polygons results in, at most, one polygon, $\Theta(n)$ polygons can stem from the intersection of two polygons, monotone with respect to the same axis. We present an algorithm for finding the intersection of two uniformly monotone polygons in linear time and space without preprocessing.

In Sec. 2, we present preliminary definitions and context. In Sec. 3, we review some facts about monotone chains and define the upper chain and lower chain of two uniformly

monotone chains. Strips are defined in Sec. 4, and an algorithm for intersecting two strips is presented. In Sec. 5, monotone polygons are presented, along with some new results. We conclude with an algorithm for intersecting two uniformly monotone polygons.

2. PRELIMINARIES

Definition 1: Let f be a function from an interval I to a point set J on the real number line. The function f is

$$\left\{ \begin{array}{l} \text{strictly increasing} \\ \text{strictly decreasing} \\ \text{nondecreasing} \\ \text{nonincreasing} \end{array} \right\} \text{ if for any } x, y \in I \text{ where } x < y, \left\{ \begin{array}{l} f(x) < f(y) \\ f(x) > f(y) \\ f(x) \leq f(y) \\ f(x) \geq f(y) \end{array} \right\}.$$

If f satisfies any of the conditions above, then f is a monotone function. If f is either strictly increasing or strictly decreasing, then f is a strictly monotone function.

Definition 2: A parameterized chain f is a continuous piecewise linear function from an interval I on the real number line to a curve in the plane. The range of f , $f(I)$, is called the chain of f .

Figure 1 illustrates a parameterized chain f whose domain, I , is a closed subset of the real number line and whose range is a curve in E^2 .

A function f defined on a point set D is *locally one-to-one* if every $p \in D$ has a neighborhood on which f is one-to-one.

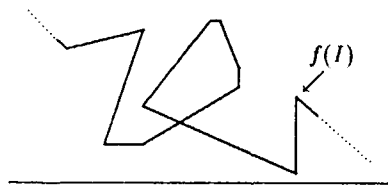


Fig. 1—Parameterized chain f defined on the interval I

3. MONOTONE CHAINS

A monotone parameterized chain is a special type of parameterized chain whose projection onto at least one axis (not necessarily the x -axis or y -axis) is either nondecreasing or nonincreasing. Consider the points along an axis ℓ as having a standard order, like that of the real number line. The projection of a point or interval α , in E^2 , onto ℓ is denoted by $\pi(\alpha, \ell)$; however, when no ambiguity will result, the abbreviation $\pi(\alpha)$ is used.

Definition 3: A parameterized chain f is monotone with respect to an axis ℓ if the function $\pi(f(x))$ is monotone.

Proposition 1: Let f be a continuous one-to-one function defined on a compact set; then f inverse, f^{-1} , is continuous.

Proposition 2: If f is a continuous function from S to R and g is a continuous function from D to S , then the composition of f and g , $f \circ g$ is also continuous.

Proposition 3: If g is a one-to-one function defined on a set S and f is a one-to-one function defined on $g(S)$, then $f \circ g$ is also one-to-one.

Proposition 4: If f is a real-valued, one-to-one, continuous function defined on the interval I , then f is monotone.

PROOF. If f is not monotone, then there exist three points $a, b, c \in I : a < b < c$ where either $f(b)$ is greater than both $f(a)$ and $f(c)$ or $f(b)$ is less than both. Consider the first case and suppose $f(a) < f(c) < f(b)$. Then by the mean-value theorem, there is $x \in (a, b)$ such that $f(x) = f(c)$. But f is one-to-one, so $x = c$, a contradiction.

Likewise, $f(b) < f(a)$ and $f(b) < f(c)$ leads to a similar conflict. □

Lemma 1: If f and g are locally one-to-one parameterized chains defined on the intervals I and J , respectively, and if $f(I) = g(J)$, then g is monotone with respect to an axis ℓ if, and only if, f is monotone with respect to ℓ .

PROOF. Suppose f is monotone with respect to ℓ . First consider the case where both f and g are one-to-one. By Proposition 1, f^{-1} is continuous. There exists a function

$$m = f^{-1} \circ g$$

which is a continuous one-to-one mapping from J to I by Propositions 2 and 3. Thus, for any $p \in I$, there is a $p' \in J$ such that $f(p) = g(p')$. Furthermore, Proposition 4 affirms that m is monotone.

If g is not monotone with respect to ℓ , then there exist three points $a, b, c \in J : a < b < c$ where either

$$\pi(g(b)) > \pi(g(a)) \quad \text{and} \quad \pi(g(b)) > \pi(g(c))$$

or

$$\pi(g(b)) < \pi(g(a)) \quad \text{and} \quad \pi(g(b)) < \pi(g(c))$$

Consider, for the moment, the first case. In I there are exactly three points, a', b' , and c' such that

$$f(a') = g(a), \quad f(b') = g(b), \quad \text{and} \quad f(c') = g(c).$$

If f is nondecreasing, then $b' > a'$ and $b' > c'$ which suggests $m((a, b)) \cap m((b, c)) \neq \emptyset$. Suppose $z \in (a', b') \cap (c', b')$; then by the mean-value theorem, there exists $x \in (a, b)$ and $y \in (c, b)$ such that $m(x) = m(y) = z$. But m is one-to-one, so $x = y$, a contradiction. Alternatively, if f is nonincreasing, then $b' < a'$ and $b' < c'$ leads to another contradiction.

An argument similar to the one used in the preceding paragraph shows the impossibility of $\pi(g(b)) < \pi(g(a))$ and $\pi(g(b)) < \pi(g(c))$. Therefore, if f is monotone with respect to ℓ , then g is monotone with respect to ℓ , when both are one-to-one.

Now the constraint that f be one-to-one is relaxed. Let f be locally one-to-one and suppose there exist three points $a, b, c \in I : a < b < c$ where $\pi(f(a)) = \pi(f(c)) \neq \pi(f(b))$. Then f is not monotone with respect to ℓ , a contradiction. By interchanging f and g , the proof is complete. \square

We are now able to refer to chains as being monotone. Figure 2 shows a monotone chain whose projection onto its monotone axis, i.e., the x -axis, is $[a, b]$. Chains that are monotone with respect to the same axis are called "uniformly monotone."

If C is a chain that is strictly monotone with respect to ℓ , then the mapping from $\pi(C)$ to C , $\pi_C^{-1}(x, \ell)$, is a function. The function $d(p, \ell)$ denotes the signed distance from the line ℓ to the point p ; $d(p, \ell) = |v| \sin \theta$ where $|v|$ is the norm of the vector from the origin of ℓ to p , and $\theta \in [0^\circ, 360^\circ)$ is the angle from ℓ to v in a counterclockwise direction (see Fig. 3).

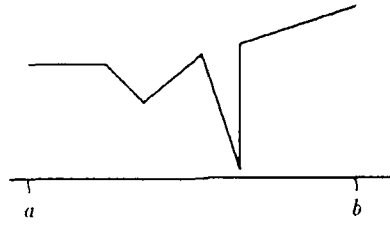


Fig. 2—A chain monotone with respect to the x -axis

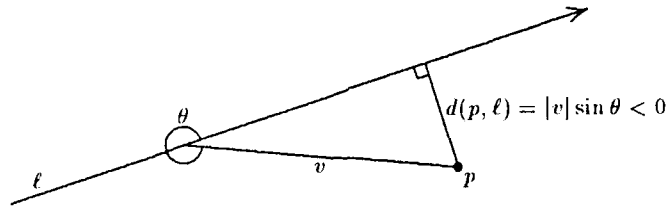


Fig. 3—A point p whose distance from axis ℓ is negative

Proposition 5: If C and D are two chains uniformly strictly monotone with respect to ℓ and $\pi(C) = \pi(D) = [a, b]$, then

$$\underline{h}(x) = \begin{cases} \pi_C^{-1}(x) & \text{if } d(\pi_C^{-1}(x), \ell) \leq d(\pi_D^{-1}(x), \ell) \\ \pi_D^{-1}(x) & \text{otherwise,} \end{cases}$$

and

$$\bar{h}(x) = \begin{cases} \pi_C^{-1}(x) & \text{if } d(\pi_C^{-1}(x), \ell) \geq d(\pi_D^{-1}(x), \ell) \\ \pi_D^{-1}(x) & \text{otherwise,} \end{cases}$$

are both parameterized chains defined on the interval $[a, b]$.

PROOF. Both $f = \pi_C^{-1}$ and $g = \pi_D^{-1}$ are parameterized chains, monotone with respect to ℓ .

First, consider \underline{h} . Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ be a partition of the interval $[a, b]$ such that either $d(f(x), \ell) \leq d(g(x), \ell)$ and $d(f(y), \ell) > d(g(y), \ell)$ or $d(f(x), \ell) \geq d(g(x), \ell)$ and $d(f(y), \ell) < d(g(y), \ell)$ for any $\lambda_{2i-1} < x \leq \lambda_{2i} < y \leq \lambda_{2i+1}$, where $i = 1, 2, \dots, (m-1)/2$. The function \underline{h} is linear on each subinterval $[\lambda_{2i}, \lambda_{2i+1}]$; thus, it is piecewise linear.

The only possible points at which \underline{h} could be discontinuous are the points in Λ . Let $\lambda \in \Lambda$ and $\varepsilon > 0$. There exists a $\delta > 0$ such that for $x \in [a, b]$, if $|\lambda - x| < \delta$, then $|f(\lambda) - f(x)| < \varepsilon$, and $|g(\lambda) - g(x)| < \varepsilon$. But $f(\lambda) = g(\lambda)$, so $|\underline{h}(\lambda) - \underline{h}(x)| < \varepsilon$.

The function $\bar{h}(x)$ can be shown to be a parameterized chain in a similar way. \square

For two strictly uniformly monotone chains C and D , the range of $\underline{h}(\bar{h})$ is the *lower chain* (*upper chain*) of C and D . However, if C or D is a monotone chain, \underline{h} and \bar{h} are undefined. The problem arises from the fact that the mappings π_C^{-1} and π_D^{-1} may not be functions. However, in the same spirit, the notions of a "lower chain" and an "upper chain" are still applicable.

Let C be a chain monotone with respect to ℓ , with $\pi(C) = I$. For an $x \in I$, define

$$m_C(x) = p : \min_{p \in \pi_C^{-1}(x)} d(p, \ell),$$

and

$$M_C(x) = p : \max_{p \in \pi_C^{-1}(x)} d(p, \ell).$$

If C is a monotone chain, then $C_{(a,b)}$ is the subchain of C whose projection onto the monotone axis is the open interval (a,b) . The subchain $C_{[a,b]}$ is a little more complicated: $C_{[a,b]} = \pi_C^{-1}(a) \cup C_{(a,b)} \cup \pi_C^{-1}(b)$. Suppose C and D are uniformly monotone and share a common subchain. The *maximal common subchain* is $C_{[a,b]} = D_{[a,b]}$: $C_{[x,a]} \neq D_{[x,a]}$ for $x < a$, and $C_{[b,y]} \neq D_{[b,y]}$ for $b < y$. For two disjoint chains, $C_{[a,b]}$ and $D_{[a,b]}$, if $d(m_C(x), \ell) > d(M_D(x), \ell)$ for any point $x \in [a,b]$, then chain $C_{[a,b]}$ is "above" $D_{[a,b]}$.

Suppose C and D are uniformly monotone with respect to ℓ with $\pi(C) \cap \pi(D) = [a, b]$, and let $a \leq \lambda_1 < \lambda_2 < \dots < \lambda_n \leq b$ be the projection onto ℓ of the points at which C and D intersect. The points of intersection are selected using the following two criteria. First, for each common subchain not orthogonal to ℓ , there is an $1 \leq i < n$ such that

$$(\pi_C^{-1}(\lambda_i) \cap \pi_D^{-1}(\lambda_i)) \cup C_{(\lambda_i, \lambda_{i+1})} \cup (\pi_C^{-1}(\lambda_{i+1}) \cap \pi_D^{-1}(\lambda_{i+1}))$$

is a maximal common subchain. And second, if the maximal common subchain is orthogonal to ℓ , then λ_i for some i is the subchain's projection onto ℓ .

The points of $C_i = C_{(\lambda_i, \lambda_{i+1})}$ have two limit points, the *left limit point* $l(C_i) : \pi(l(C_i)) = \lambda_i$ and the *right limit point* $r(C_i) : \pi(r(C_i)) = \lambda_{i+1}$. The union of C_i and its two limit points is called C_i^* for $1 \leq i < n$.

However, the extreme subchains have a different composition based on whether an upper subchain or a lower subchain is being defined. The extreme upper subchains U_0 and U_n require C_0^* and C_n^* to have the following composition:

$$\begin{aligned} C_0^* &= C_{(a, \lambda_1)} \cup \{r(C_{(a, \lambda_1)})\} \\ C_n^* &= C_{(\lambda_n, b)} \cup \{\ell(C_{(\lambda_n, b)}), r(C_{(\lambda_n, b)})\} \end{aligned}$$

whereas L_0 and L_n require a slightly different composition:

$$\begin{aligned} C_0^* &= C_{(a, \lambda_1)} \cup \{\ell(C_{(a, \lambda_1)}), r(C_{(a, \lambda_1)})\} \\ C_n^* &= C_{(\lambda_n, b)} \cup \{\ell(C_{(\lambda_n, b)})\} \end{aligned}$$

Letting $\lambda_0 = a$ and $\lambda_{n+1} = b$, for $0 \leq i \leq n$ we define

$$\begin{aligned} U_i &= \begin{cases} C_i^* & \text{if } C_{(\lambda_i, \lambda_{i+1})} \text{ is above } D_{(\lambda_i, \lambda_{i+1})} \\ D_i^* & \text{otherwise} \end{cases} \\ L_i &= \begin{cases} D_i^* & \text{if } C_{(\lambda_i, \lambda_{i+1})} \text{ is above } D_{(\lambda_i, \lambda_{i+1})} \\ C_i^* & \text{otherwise} \end{cases} \end{aligned}$$

Definition 4: Suppose C and D are chains uniformly monotone with respect to ℓ , with $\pi(C) \cap \pi(D) = [a, b]$, and let $a \leq \lambda_1 < \dots < \lambda_n \leq b$ be the projection onto ℓ of the points of intersection. Then the upper chain is the set of points in

$$U = U_0 \cup \left(\bigcup_{i=1}^n I_i \cup U_i \right)$$

where $I_i = [\min_{d(p, \ell)} \{r(U_{i-1}), \ell(U_i)\}, \max_{d(p, \ell)} \{r(U_{i-1}), \ell(U_i)\}]$. Similarly, the lower chain is the union

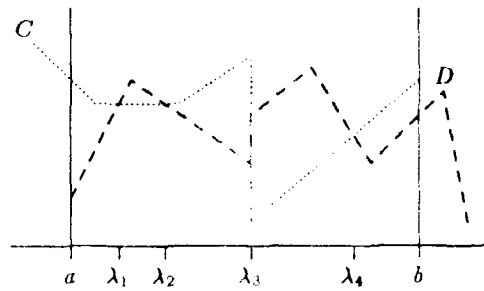
$$L = L_0 \cup \left(\bigcup_{i=1}^n I_i \cup L_i \right)$$

where $I_i = [\min_{d(p, \ell)} \{r(L_{i-1}), \ell(L_i)\}, \max_{d(p, \ell)} \{r(L_{i-1}), \ell(L_i)\}]$.

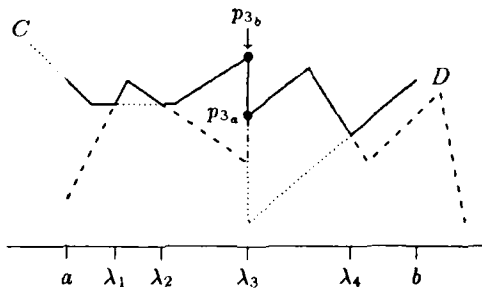
Figure 4 depicts two chains, C (dotted) and D (dashed), uniformly monotone with respect to the x -axis. In Fig. 4(b), the upper chain of C and D is a solid line.

An important fact of uniformly monotone chains is that their upper and lower chains are also monotone.

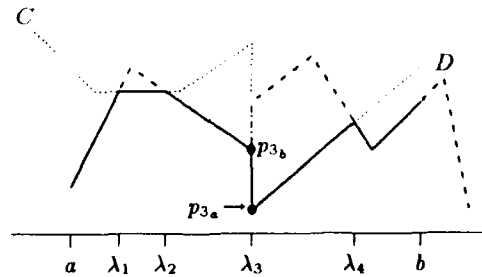
Lemma 2: *If C and D are chains, uniformly monotone with respect to the axis ℓ , then both the upper chain and lower chain of C and D are monotone with respect to ℓ .*



(a): Chain C and chain D



(b): Upper chain of C and D



(c): Lower chain of C and D

Fig. 4—The upper chain and lower chain of two chains uniformly monotone with respect to the x -axis

4. STRIPS

Definition 5: Let C be a chain monotone with respect to ℓ , where $\pi(C) = [a, b]$. The space bounded between the two lines perpendicular to ℓ , one passing through a , the other through b , is divided into two regions—the subspace strictly above C , $strip^+(C, \ell)$ and the subspace strictly below C , $strip^-(C, \ell)$. If no confusion will result, $strip^+(C, \ell)$ and $strip^-(C, \ell)$ are shortened to $strip^+(C)$ and $strip^-(C)$, respectively. One rationale for using the words “upper” and “lower” in the terms upper chain and lower chain can be explained using strips. Suppose C and D are two uniformly monotone chains, and let U and L be the upper chain and lower chain of C and D . Then, there does not exist a point in $C \cup D$ which is also in $strip^+(U)$ or in $strip^-(L)$.

Proposition 6: Let U and L be chains uniformly monotone with respect to ℓ . A point $p : \pi(p) \in \pi(L) \cap \pi(U)$ is in $strip^+(L) \cap strip^-(U)$ if and only if

$$d(M_L(\pi(p)), \ell) < d(p, \ell) < d(m_U(\pi(p)), \ell).$$

Figure 5 illustrates the intersection of two strips, $strip^+(L)$ and $strip^-(U)$. Both chains L and U are monotone with respect to the x -axis. Notice that the $strip^+(L) \cap strip^-(U)$ is composed of two polygons. One polygon is the union of U_1 and L_1 , whereas the other polygon is the union of U_3 and L_3 .

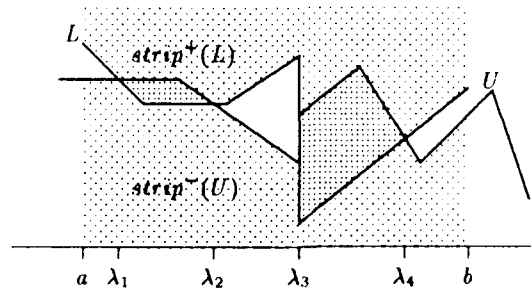


Fig. 5— $Strip^+(L, \ell) \cap strip^-(U, \ell)$

STRIP INTERSECTION

Let C and D be two uniformly monotone chains with respect to ℓ , with $\pi(C) \cap \pi(D) = [a, b]$ and let $a \leq \lambda_1 < \dots < \lambda_{n-1} \leq \lambda_n = b$ be the projection onto ℓ of the points of intersection. The upper chain and lower chain of C and D are U and L , respectively.

The function STRIP-INTERSECTION(C, D, ℓ) returns a set of polygons \mathcal{P} which form the boundary of $strip^+(L) \cap strip^-(U)$.

PARAMETER AND LOCAL VARIABLE DESCRIPTIONS

- C A chain, monotone with respect to ℓ
- D A chain, monotone with respect to ℓ
- ℓ The monotone axis for C, D and all polygons in \mathcal{P}
- \mathcal{P} The set of polygons (possibly empty) which forms the boundary of $strip^+(L) \cap strip^-(U)$
- L The lower chain of C and D
- U The upper chain of C and D
- n The index of the last subchain L_n, U_n of L, U , respectively.

STRIP-INTERSECTION(C, D, ℓ)

```
begin
1.   $\mathcal{P} := \{ \}$ 
2.   $i := 1$ 
3.  while ( $i < n$ ) do
      begin
4.      if ( $U_i$  is above  $L_i$ ) then
          begin
5.           $P_i := [l(L_i), l(U_i)] \cup (U_i)_{(\lambda_i, \lambda_{i+1})}$ 
               $\cup (L_i)_{(\lambda_i, \lambda_{i+1})} \cup [r(L_i), r(U_i)]$ 
6.           $\mathcal{P} := \mathcal{P} \cup P_i$ 
          end
      end
7.       $i := i + 1$ 
      end
8.  return( $\mathcal{P}$ )
end
```

5. MONOTONE POLYGONS

In general, a polygon separates the plane into two regions, its "interior" (the bounded region) and its "exterior" (the unbounded region), by the Jordan Curve Theorem. The polygon is the "boundary" of its interior.

The path formed by moving along a polygon in a counterclockwise (clockwise) direction about its interior is said to have a counterclockwise (clockwise) "winding."

Definition 6: A (strictly) monotone polygon P is a chain, the union of two (strictly) uniformly monotone subchains L_P and U_P ; L_P , U_P are the lower chain, upper chain of L_P and U_P , respectively.

Note that Definition 6 requires $\pi(L_P) = \pi(U_P)$. Two polygons (strictly) monotone with respect to the same axis are called *uniformly (strictly) monotone* (Lui and Ntafos, 1988).

Figure 6 shows a strictly monotone polygon with respect to the x -axis.

The next theorem is the foundation for our monotone polygon intersection algorithm.

Theorem 1: Let P and Q be two uniformly monotone polygons, let U be the upper chain of L_P and L_Q , and let L be the lower chain of U_P and U_Q . Then, a point $p : \pi(p) \in \pi(P) \cap \pi(Q)$ is interior to $P \cap Q$ if and only if

$$p \in \text{strip}^-(L) \cap \text{strip}^+(U).$$

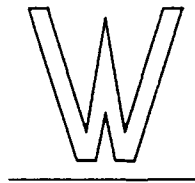


Fig. 6—A polygon monotone with respect to the x -axis

PROOF. Suppose p is interior to $P \cap Q$; then

$$d(p, \ell) > d(M_{LP}(\pi(p)), \ell)$$

and

$$d(p, \ell) > d(M_{LQ}(\pi(p)), \ell)$$

Thus, $p \in \text{strip}^+(U)$. In a similar vein, $p \in \text{strip}^-(L)$.

Conversely, suppose p is interior to $\text{strip}^-(L) \cap \text{strip}^+(U)$; then

$$d(M_{LP}(\pi(p)), \ell) < d(p, \ell) < d(m_{UP}(\pi(p)), \ell),$$

which establishes p is interior to P . Substituting Q for P affirms p must also be interior to Q . □

Part (a) of Fig. 7 portrays the intersection of two polygons, a post P and a serpent Q , uniformly monotone with respect to the x -axis. In (b) of Fig. 7, the upper chain of the polygons' lower chains is drawn as a solid line; likewise, in (c) the lower chain of the polygons' upper chains is a solid line. Finally, (d) of Fig. 7 visually illustrates Theorem 1.

Polygon intersection is loosely used to connote the polygon(s) that form the boundary of the intersection of the polygons' interiors. Thus, any polygon formed by the intersection of two uniformly monotone polygons is also monotone with respect to the same axis. The polygon is the union of the upper chain of uniformly monotone subchains and of the lower chain of uniformly monotone subchains.

Corollary 1: *If P and Q are uniformly monotone polygons with respect to the axis ℓ , then the polygons formed by intersecting P and Q are all uniformly monotone with respect to ℓ .*

MONOTONE POLYGON INTERSECTION ALGORITHM

Let P and Q be polygons uniformly monotone with respect to ℓ .¹ The algorithm $\text{MONOTONE-POLYGON-INTERSECTION}(P, Q, \ell)$ returns a set of polygons $\mathcal{P} = P \cap Q$.

¹If P and Q are convex polygons, then ℓ can be any arbitrary line.

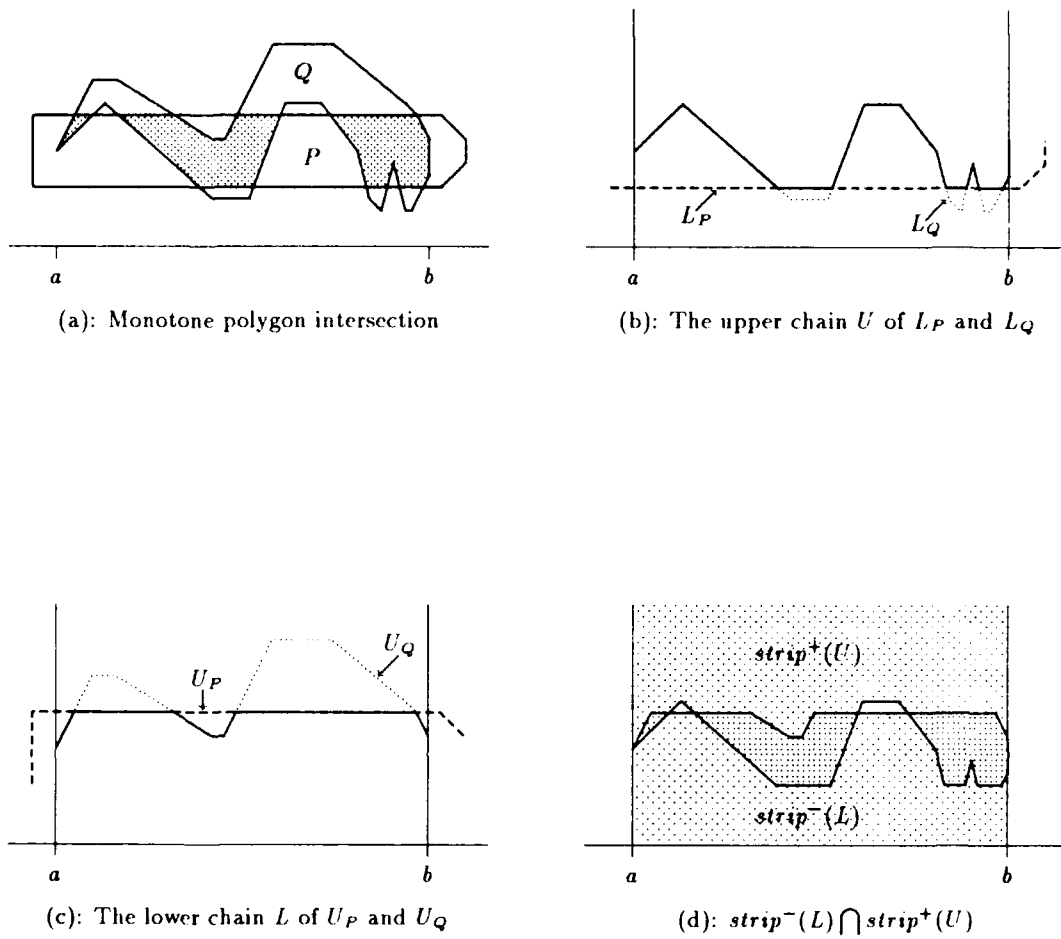


Fig. 7—The intersection of two polygons uniformly monotone with respect to the x -axis

PARAMETER AND LOCAL VARIABLE DESCRIPTIONS

- P A polygon, monotone with respect to ℓ
 Q A polygon, monotone with respect to ℓ
 ℓ The axis from which both P and Q are monotone
 U_P The upper chain of P
 L_P The lower chain of P
 U_Q The upper chain of Q
 L_Q The lower chain of Q
 U The upper chain of L_P and L_Q
 L The lower chain of U_P and U_Q
 \mathcal{P} The set of polygons forming $P \cap Q$

MONOTONE-POLYGON-INTERSECTION(P, Q, ℓ)

```
begin
1.  if  $(\pi(P) \cap \pi(Q) = \emptyset)$  or  $(L_P$  is above  $U_Q)$  or  $(L_Q$  is above  $U_P)$  then
2.     $\mathcal{P} := \emptyset$ 
3.  else if  $U = U_P$  and  $L = L_P$  then  $\mathcal{P} := P$ 
4.  else if  $U = U_Q$  and  $L = L_Q$  then  $\mathcal{P} := Q$ 
5.  else  $\mathcal{P} := \text{STRIP-INTERSECTION}(U, L, \ell)$ 
6.  return( $\mathcal{P}$ )
end
```

Analyzing the algorithm's time-complexity is straightforward. The upper and lower chains of a monotone polygon can be found by examining all vertices once. Computing U and L possibly requires another loop over both polygons' vertices. The conditions in lines 1, 3, and 4 can be answered in $\Theta(1)$ time by considering these cases while U and L are being constructed. Finally, the function STRIP-INTERSECTION visits the polygons' vertices once at most. We conclude that two uniformly monotone polygons can be intersected by visiting all vertices at most three times.

The two functions we have presented require the monotone axis ℓ as an argument. In fact, finding the axes for which a simple polygon is monotone requires $\Theta(n)$ time (Preparata and Supowit, 1981). Therefore, the intersection of two polygons both monotone with respect to some axis can still be found in linear time, without the axis known a priori.

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